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LETTER

Synchronization of canards in resistively and capacitively coupled canard-generating Bonhoeffer–van der Pol oscillators

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SUMMARY This letter presents an investigation into the synchronization of an autonomous system comprising two nearly identical canard-generating Bonhoeffer–van der Pol (BVP) oscillators coupled via a resistor and a capacitor in parallel. We first demonstrate via numerical simulations that this system exhibits butterfly synchronization, i.e., a phase shift between the canards in a weakly coupled system of nearly identical canard-generating BVP oscillators. Furthermore, the butterfly synchronization in the coupled system is observed experimentally.
key words: *Synchronization, canard, Bonhoeffer–van der Pol oscillator,*

1. Introduction

Synchronization is a widespread phenomenon in nature, science, and engineering [1]. In this letter, we numerically investigate the occurrence of a special type of synchronization that is referred to as "butterfly synchronization" [9] in the autonomous system consisting of two nearly identical canard-generating Bonhoeffer–van der Pol (BVP) oscillators coupled via a resistor and a capacitor in parallel. In addition, we verify the occurrence of this synchronization via experimental work.

In recent years, high-dimensional autonomous systems that generate canards have attracted considerable research interest [2]–[9]. In such systems, the synchronization of canards can be observed. In previous works, the complete and in-phase synchronization of canards in identical and nearly identical canard-generating BVP oscillators coupled via a resistor have been investigated [8], [9]. It has been found that both complete and in-phase synchronization can be observed across a wide range of coupling parameter values. However, in experimental work, it was found that both the complete and in-phase synchronizations break when the coupling parameter is even slightly decreased from the value at which complete synchronization is observed. At lower values of the coupling parameter, a special synchronization phenomenon, which was termed "butterfly synchronization", was observed [9].

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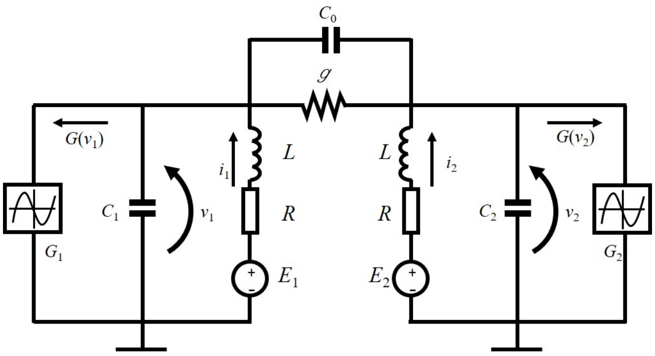

Fig. 1 Two BVP oscillators coupled via a resistor and a capacitor in parallel.

It was hypothesized that the appearance of butterfly synchronization could be due to the noise in the electric circuit. A non-autonomous system with two sinusoidal perturbations with a moderate phase difference was then investigated, and it was found that butterfly synchronization could be observed numerically in this non-autonomous system that was subject to perturbations [9]. This letter presents another system in which butterfly synchronization can be observed numerically. Here, we propose a system comprising two BVP oscillators coupled via a capacitor and a resistor in parallel (see Fig. 1). The capacitor in the system considered in this study is of interest because we hypothesize that such a circuit inherently exhibits stray capacitance; it is thus of interest to study the behavior of a system containing this coupling element. The inherent stray capacitance in the circuit may be sufficient to explain the discrepancy between the results of numerical and experimental results observed in previous work [9]. We find that butterfly synchronization can occur in our proposed autonomous system when the system is characterized by a low coupling conductance and a low coupling capacitance. We also find that complete synchronization is seen when the system is characterized by a high coupling conductance. In addition, we successfully observe butterfly synchronization experimentally.

2. Numerical analysis of the resistively and capacitively coupled BVP oscillators with a mismatch in capacitances

In this section, we discuss the synchronization of canards that are generated by nearly identical coupled BVP oscillators. Figure 1 shows the circuit diagram of the system that we propose in this study; this system consists of two-coupled BVP oscillators with a conductance g and a capacitance C_0. We introduce a slight parameter mismatch between the two oscillators to avoid an unintended complete synchronization that may occur as a result of the finite precision of the computations in the absence of such a mismatch. Furthermore, since it is difficult to construct qualitatively similar elements with nonlinear conductance using diode arrays, a 10% parameter mismatch in the two capacitances C_1 and C_2 is introduced. This mismatch permits the results of the numerical investigations to accurately represent the results obtained experimentally.

The governing equations of the coupled circuits can be expressed by the following system of four autonomous ordinary differential equations:

$$\begin{cases} C_1\dfrac{dv_1}{dt} = -i_{d1} + i_1 + \dfrac{(v_1 - v_2)}{R_c} - C_0\dfrac{d(v_1 - v_2)}{dt}, \\ L\dfrac{di_1}{dt} = E_1 - v_1 - i_1R, \\ C_2\dfrac{dv_2}{dt} = -i_{d2} + i_2 + \dfrac{(v_1 - v_2)}{R_c} + C_0\dfrac{d(v_1 - v_2)}{dt}, \\ L\dfrac{di_2}{dt} = E_2 - v_2 - i_2R. \end{cases} \tag{1}$$

Here, we utilize the following variable transformations:

$$\begin{aligned} &v_1 = \sqrt{\frac{g_1}{g_3}}x_1,\ v_2 = \sqrt{\frac{g_1}{g_3}}x_2,\ i_1 = g_1\sqrt{\frac{g_1}{g_3}}y_1, \\ &i_2 = g_1\sqrt{\frac{g_1}{g_3}}y_2,\ t = Lg_1\tau,\ E_1 = \sqrt{\frac{g_1}{g_3}}B_1,\ E_2 = \sqrt{\frac{g_1}{g_3}}B_2, \\ &k = g_1R,\ \varepsilon_1 = \frac{C_1}{Lg_1^2},\ \varepsilon_2 = \frac{C_2}{Lg_1^2},\ \varepsilon_0 = \frac{C_0}{Lg_1^2},\ \alpha = \frac{g}{g_1}. \end{aligned} \tag{2}$$

Using these transformations, the governing equations can be rewritten in a normalized form:

$$\begin{cases} \dot{x}_1 = \dfrac{\varepsilon_2(y_1 + x_1 - x_1^3 - \alpha(x_1 - x_2))}{\varepsilon_1\varepsilon_2 + \varepsilon_0(\varepsilon_1 + \varepsilon_2)} + \\ \quad \dfrac{\varepsilon_0(x_1 + x_2 + y_1 + y_2 - (x_1^3 + x_2^3))}{\varepsilon_1\varepsilon_2 + \varepsilon_0(\varepsilon_1 + \varepsilon_2)}, \\ \dot{y}_1 = -x_1 - ky_1 + B_1, \\ \dot{x}_2 = \dfrac{\varepsilon_1(y_2 + x_2 - x_2^3 + \alpha(x_1 - x_2))}{\varepsilon_1\varepsilon_2 + \varepsilon_0(\varepsilon_1 + \varepsilon_2)} + \\ \quad \dfrac{\varepsilon_0(x_1 + x_2 + y_1 + y_2 - (x_1^3 + x_2^3))}{\varepsilon_1\varepsilon_2 + \varepsilon_0(\varepsilon_1 + \varepsilon_2)}, \\ \dot{y}_2 = -x_2 - ky_2 + B_2. \end{cases} \tag{3}$$

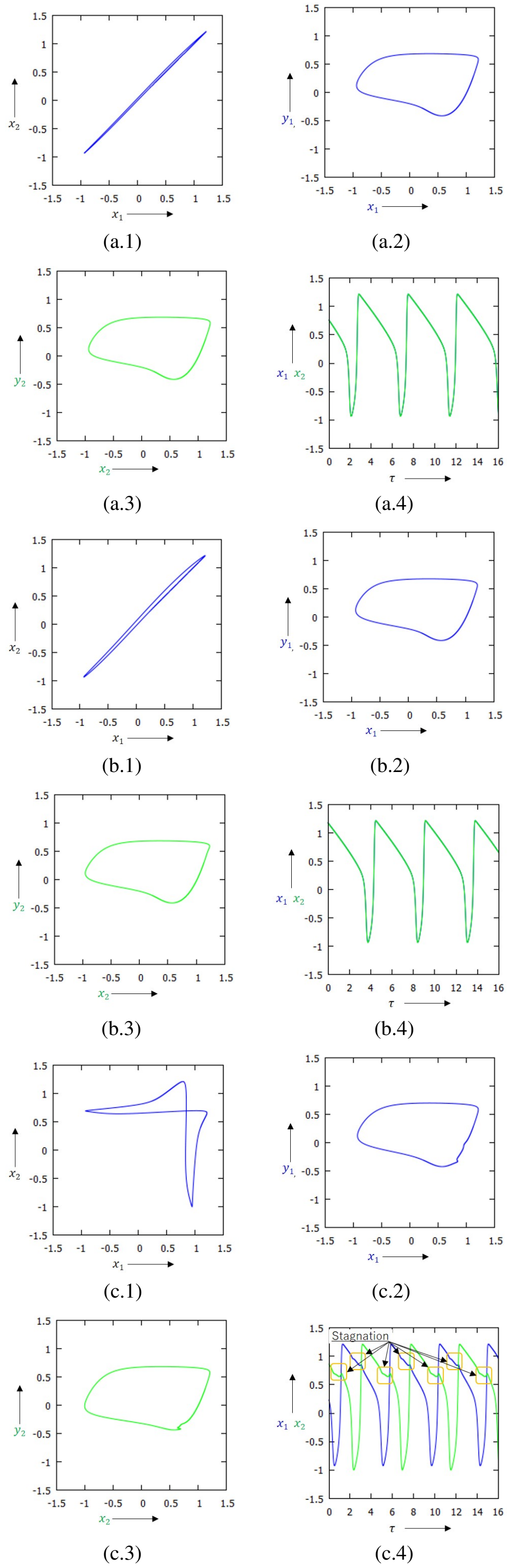

Fig. 2 (a) Numerical results showing the almost-complete (pseudo-complete) synchronization of canards for $\alpha = 1.2$. Trajectories projected onto the (a.1) x_1–x_2, (a.2) x_1–y_1, and (a.3) x_2–y_2 planes; (a.4) the time-series waveform for x_1 (blue) and x_2 (green). (b) Numerical results showing the phase shift of canards for $\alpha = 0.5$. Trajectories projected onto the (b.1) x_1–x_2, (b.2) x_1–y_1, and (b.3) x_2–y_2 planes; (b.4) the time-series wave-forms for x_1 (blue) and x_2 (green). (c) Numerical results showing the butterfly synchronization of canards for $\alpha = 0.01$. Trajectories projected onto the (c.1) x_1–x_2, (c.2) x_1–y_1, and (c.3) x_2–y_2 planes; (c.4) the time-series waveforms for x_1 (blue) and x_2 (green) of the autonomous system for ε_0=0.01.

Here, B_1 and B_2, which are the parameters that contain information related to the DC bias voltages E_1 and E_2, respectively, were both set to 0.4897. This parameter value was found to yield a canard in both oscillators. The parameters ε_1 and ε_2 describe the capacitors within the BVP oscillators. As mentioned above, a 10% parameter mismatch was introduced in the capacitances of the capacitors in the BVP oscillators; the actual values of the parameters describing the capacitors in the system in this work were ε_1 = 0.1, ε_2 = 0.09, and ε_0= 0.01. We note that the values of these parameters (B_1, B_2, ε_0, ε_1, and ε_2) are the same as those used in Ref. [9]. Furthermore, we note that α is a parameter that corresponds to the conductance g; the parameter g is used as the bifurcation parameter in this section. We also note that ε_0 corresponds to the coupling capacitance C_0.

For large values of α (for example, α = 1.2), the system comprising two BVP oscillators coupled via a resistor and a capacitor in parallel exhibits almost-complete synchro-nization, as shown in Fig. 2(a). The trajectory in the x_i–y_i plane for i = 1, 2 shows a canard. However, for very small values of α (for example, α = 0.01), the phase difference between the two oscillators increases. As a result, butterfly synchronization occurs, as shown in Fig. 2(c). We also note that in the time-series of this system (with the capacitive coupling element), at this smaller value of α, the attractor is not smooth as observed in our previous work [9] and we notice that there are areas where the stagnation occurred as indicated by the arrows in Fig. 2(c.4). This may be due to the effect of the coupling capacitor.

In addition, for intermediate values of α (for example, α = 0.5), the phase shift between the two oscillators appear as shown in Fig. 2(b) where the x_1–x_2 plot starts to bloat and eventually results in butterfly synchronization after the value of α is further decreased to 0.01 as described above.

In the next section, we present the results of circuit experiments on the system proposed here in order to verify the results of the numerical analysis.

3. Experimental study of two nearly identical canard-generating coupled oscillators

In the circuit experiment, we first investigate the properties of the system for large values of the coupling conductance g. The capacitor used to couple the two BVP oscillators (placed in parallel with the resistor) has a capacitance of 10 pF. Similar to previous works [9], for large values of g, when $g \simeq 445\mu$S ($\alpha \simeq 1.2$), the oscillators exhibit an almost-complete (pseudo-complete) synchronization state, as shown in Fig. 3(a.1). Figures 3(a.2) and (a.3) show the attractors of the canard shape projected onto the phase plane.

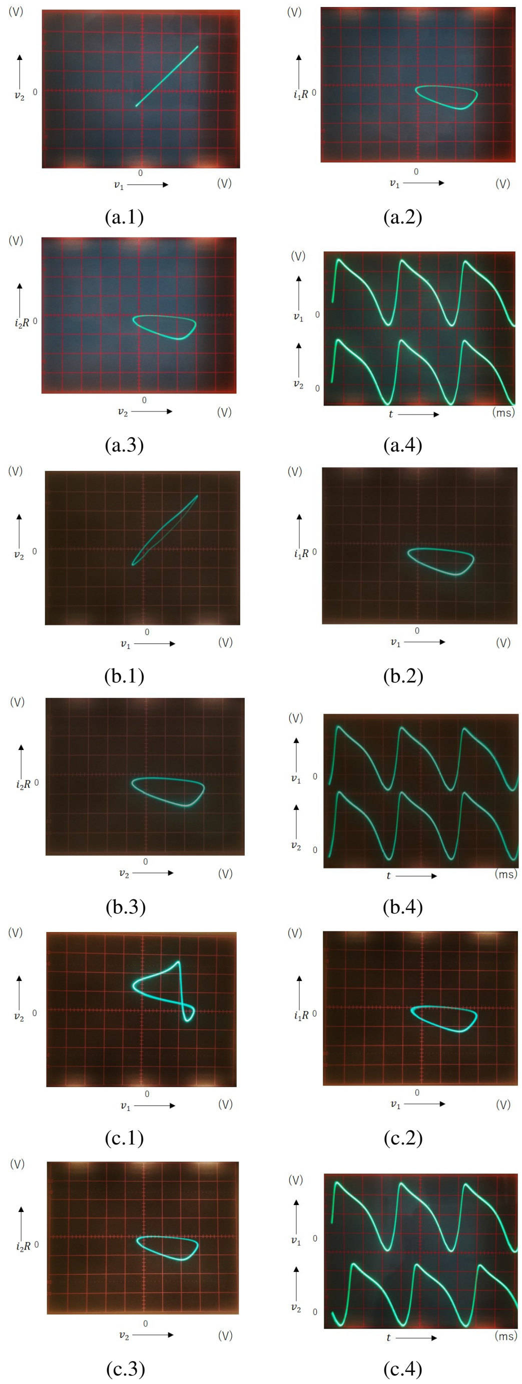

Fig. 3 Experimental measurements of the synchronized canards. The almost-complete (pseudo-complete) synchronization in the planes (a.1) v_1–v_2, (a.2) v_1–i_1R, and (a.3) v_2–i_2R; (a.4) the time-series waveforms of v_1 (upper trace) and v_2 (lower trace) in a system with a coupling of $g = 445\mu$S and a small coupling capacitance of C_0 = 10 pF. Phase shift of canards in the planes (b.1) v_1–v_2, (b.2) v_1–i_1R, and (b.3) v_2–i_2R; (b.4) the time-series waveforms of v_1 (upper trace) and v_2 (lower trace) in a system with a coupling of $g = 100\mu$S and a small coupling capacitance of C_0 = 10 pF. The butterfly synchronization in the planes (c.1) v_1–v_2, (c.2) v_1–i_1R, and (c.3) v_2–i_2R; (c.4) the time-series waveforms of v_1 (upper trace) and v_2 (lower trace) in a system with coupling described by a low conductance g = 0.425μS and a small coupling capacitance C_0 = 10 pF. The grid meshes represent 0.5 V/div in both the horizontal and vertical directions in figures (a.1), (a.2), (a.3), (b.1), (b.2), (b.3), (c.1), (c.2), and (c.3); the grid meshes represent 0.5 V/div and 100 µs/div in the vertical and horizontal directions, respectively, in figures (a.4). (b4), and (c.4).

When the coupling parameter α, which represents the coupling conductance, is decreased to a very small value (for example, 0.425μS) in the presence of a coupling capacitor, the phase difference between v_1 and v_2 can be clearly observed. At such a small conductance, the in-phase synchronization of the two oscillators collapses, as shown in Fig. 3(c.1). Due to the shape of the Lissajous diagram of the synchronized attractor projected onto the v_1–v_2 plane, this complex synchronization behavior has been referred to as the "butterfly synchronization" of canards [9]; this name reflects the shape of the attractors observed in the system. Our experimental results obtained here are in qualitatively agreement with the results obtained from the numerical simulations.

Also, for intermediate value of coupling conductance of $g \simeq 100\mu$S, the oscillators exhibit a phase shift, as shown in Fig. 3(b.1). Figures 3(b.2) and (b.3) show the attractors of the canard shape projected onto the phase plane. Behavior for the intermediate value of conductance is in agreement with the numerical simulation where the v_1–v_2 plot starts to bloat indicating the phase shift occurrence which eventually results in butterfly synchronization upon further decrease in the coupling conductance as discussed above.

Conclusion

In this study, we discussed the synchronization of an autonomous system consisting of two nearly identical canard-generating BVP oscillators coupled via a resistor and a capacitor in parallel. We demonstrated via numerical simulations that butterfly synchronization can be observed in an autonomous system comprising nearly identical canard-generating BVP oscillators for small values of the parameter corresponding to the coupling conductance (here, $\alpha = 0.01$) and small values of the coupling capacitance (here, $\varepsilon_0 = 0.01$). Additionally, we carried out circuit experiments and observed butterfly synchronization in two BVP oscillators coupled via a resistor with a high resistance and a capacitor with a low capacitance in parallel. It will be of interest to investigate the bifurcation phenomena that occur when the coupling strength is varied in the system proposed here.

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