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A Construction of Binary Cross Z-Complementary Pairs with Large CZC Ratio*

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SUMMARY Cross Z-complementary pairs (CZCPs), characterized by two symmetric zero autocorrelation zones (ZACZs) and one tail-end zero cross-correlation zone (ZCCZ), play an instrumental role in the design of training sequences for broadband spatial modulation systems. In this letter, we propose a systematic construction of CZCPs with large cross Z-complementary ratio (CZCR) by employing Turyn's method to some seed CZCPs and Golay complementary pairs (GCPs). By appropriately selecting the seed CZCPs, we can extend the CZCPs with parameters (18,7) and (22,9) to new $(18N, 8N-1)$-CZCPs and $(22N, 9N+Z_1)$-CZCPs, where Z_1 signifies the zero correlation zone width achievable by a binary GCP. Additionally, we introduce new CZCPs with parameters (34,14) and (38,14), which were not previously reported in the literature, and extend them to $(34N, 14N+Z_1)$-CZCPs and $(38N, 15N-1)$-CZCPs.

key words: *Cross Z-complementary pairs, Turyn's method, Golay complementary pairs, spatial modulation (SM).*

1. Introduction

Spatial modulation (SM) stands out as a unique multiple-input multiple-output (MIMO) technology, distinguished by having multiple transmit antenna (TA) elements while employing a single radio-frequency (RF) chain. This setup allows SM to reduce the overall number of RF chains required in MIMO systems, addressing some of the traditional challenges. However, SM faces a specific issue: the "one-RF-chain" principle of SM restricts the transmitter's ability to employ pilot transmission across all transmitting antennas simultaneously, but this also leads to dense training sequences for traditional MIMO systems incompatible with SM systems [1]. To tackle this issue, Liu et al. [2] proposed a novel complementary sequence, i.e., cross Z-complementary sequence pairs (CZCPs). In frequency-selective channels, the "front-end ZACZ" property of CZCPs can mitigate the intersymbol interference for each TA. Furthermore, the "tail-end ZACZ" and "tail-end ZCCZ" properties of CZCPs can mitigate inter-channel interference caused by multi-path propagation. As a result, CZCPs can be employed to generate an SM training matrix, which can achieve optimal channel estimation performance. The zero correlation zone (ZCZ) width for any CZCP of length N is less than or equal to $N/2$. If it reaches $N/2$, it is called perfect CZCP. Perfect CZCPs are special cases of Golay complementary pairs (GCPs), also known as strengthened GCPs. Liu et al. [2] constructed q-ary perfect CZCPs with parameters $(2N, N)$ and $(2^m, 2^{m-1})(m \geq 1)$ respectively by using concatenation and generalized Boolean function (GBF), where $N = 2^\alpha 10^\beta 26^\gamma$.

In 2020, Fan et al. [3] constructed binary CZCPs of lengths $10^{\beta+1}$, $26^{\gamma+1}$ and $10^\beta 26^{\gamma+1}$, having the ZCZ width $4\times 10^\beta$, $12\times 26^\gamma$ and $12\times 10^\beta 26^\gamma$, respectively. Additionally, Adhikary et al. [4] had made the following contributions: Firstly, they designed q-ary CZCPs using GBF and binary CZCPs using insertion function. Secondly, they proposed the cross Z-complementary ratio (CZCR) as a new metric to evaluate the optimality of sequence pairs. A CZCP is deemed optimal when its CZCR equals 1. Finally, they also presented the optimal (12,5) and (24,11)-CZCP by using Barker sequences. In 2021, Yang et al. [5] presented binary and quaternary CZCPs, the maximum CZCR achieved by these resultant CZCPs is approximately 6/7. Huang et al. [6] presented binary $(2^{m-1} + 2^{v+1}, 2^{\pi(v+1)-1} + 2^v - 1)$-CZCPs $(m \geq 4, 0 \leq v \leq m-3)$ with CZCR approximately 2/3 using Boolean function. Based on the resultant CZCPs of [6], Das et al. [7] obtained the corresponding q-ary CZCPs using GBFs. In 2022, Zeng et al. [8] introduced eight novel constructions of QCZCPs, encompassing lengths of $3N$, $7N$, $9N$, $11N$, $12N$, $14N$, $18N$ and $24N$, where $N = 2^\alpha 10^\beta 26^\gamma$. The maximum CZCR achieved by these CZCPs is 11/12. In 2023, Zhang et al. [9] proposed a family of CZCPs, including $(28N, 13N)$, $(48N, 23N)$, $(56N, 27N)$, $(96N, 47N)$ and $(112N, 55N)$ through Turyn's method.

Motivated by [4], [9], we will construct CZCPs with large CZCR by applying Turyn's method to certain seed CZCPs and GCPs. By appropriately selecting seed CZCPs, we can extend the known $(18, 7)$-CZCP and $(22, 9)$-CZCP to $(18N, 8N-1)$-CZCPs and $(22N, 9N+Z_1)$-CZCPs, where Z_1 represents the ZCZ width achievable by a binary GCP of length N. Specifically, when $N = 2^{\alpha+1} 10^\beta 26^\gamma$, our CZCPs reach CZCRs of approximately 8/9 and 19/22, outperforming CZCPs from [4, Th.6] having CZCRs of approximately 7/9 and 9/11. Furthermore, we present new CZCPs with parameters (34,14) and (38,14), and extend them to $(34N, 14N+Z_1)$-CZCPs and $(38N, 15N-1)$-CZCPs, respectively.

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2. Basic Definitions

The binary sequence mentioned throughout this paper refers to a sequence defined over $\mathcal{A}_2 = \{+1, -1\}$, which are represented by + and − respectively. Also, $\overleftarrow{\mathbf{a}}$ indicates the reversal of $\mathbf{a}$, $\otimes$ indicates the Kronecker product.

Definition 1: The aperiodic cross-correlation function (ACCF) of two binary sequences $\mathbf{a}$ and $\mathbf{b}$ for time-shift τ is defined as

$$R_{\mathbf{a},\mathbf{b}}(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} a_i b_{i+\tau}, & 0 \le \tau \le N-1, \\ \sum_{i=0}^{N-1+\tau} a_{i-\tau} b_i, & -(N-1) \le \tau \le -1, \\ 0, & \text{otherwise.} \end{cases}$$

Specifically, if $\mathbf{a} = \mathbf{b}$, $R_{\mathbf{a},\mathbf{b}}(\tau)$ becomes the aperiodic auto-correlation function (AACF) of $\mathbf{a}$ at time-shift τ, denoted as $R_{\mathbf{a}}(\tau)$.

Lemma 1 ([9]): Let $\mathbf{a}$ and $\mathbf{b}$ be two binary sequences of length N. Then $R_{\mathbf{a},\mathbf{b}}(\tau) = R_{\mathbf{b},\mathbf{a}}(-\tau)$, $R_{\mathbf{a}}(\tau) = R_{\overleftarrow{\mathbf{a}}}(\tau)$ and $R_{\mathbf{a},\overleftarrow{\mathbf{b}}}(\tau) = R_{\mathbf{b},\overleftarrow{\mathbf{a}}}(\tau)$, where $0 \le \tau \le N-1$.

Definition 2: A sequence pair ($\mathbf{a}$, $\mathbf{b}$) of length N is known as Golay complementary pair (GCP) if

$$R_{\mathbf{a}}(\tau) + R_{\mathbf{b}}(\tau) = 0, \ \ 1 \le \tau \le N-1.$$

Definition 3 ([2]): For an integer $Z \le M$, a pair of binary sequences $\mathbf{c}$ and $\mathbf{d}$ of length M (M even) is called cross Z-complementary pair (CZCP), briefly written as (M, Z)-CZCP, satisfying

$$\begin{aligned} &\mathrm{C1}: R_{\mathbf{c}}(\tau) + R_{\mathbf{d}}(\tau) = 0, \qquad |\tau| \in \mathcal{T}_1 \cup \mathcal{T}_2, \\ &\mathrm{C2}: R_{\mathbf{c},\mathbf{d}}(\tau) + R_{\mathbf{d},\mathbf{c}}(\tau) = 0, \qquad |\tau| \in \mathcal{T}_2, \end{aligned}$$

where $\mathcal{T}_1 = \{1, 2, \ldots, Z\}$ and $\mathcal{T}_2 = \{M-Z, M-Z+1, \ldots, M-1\}$. It is evident that $Z \le M/2$. When $Z = M/2$, the CZCP is known as perfect CZCP or strengthened GCP.

Lemma 2 ([2]): Let $(\mathbf{c}, \mathbf{d})$ be a binary (M, Z)-CZCP. Then

1. $c_i = \frac{c_0}{d_0} d_i$ and $c_{M-1-i} = -\frac{c_0}{d_0} d_{M-1-i}$ for $0 \le i < Z$.
2. $(c_1\mathbf{c}, c_2\mathbf{d})$, $(c_1\mathbf{d},\ c_2\mathbf{c})$, $(c_1\overleftarrow{\mathbf{c}}, c_2\overleftarrow{\mathbf{d}})$ are also (M, Z)-CZCPs, where $c_1, c_2 \in \{1, -1\}$.
3. $R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(\tau) - R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(\tau) = 0$ for $M-Z \le |\tau| \le M-1$.

Definition 4: Let $(\mathbf{c}, \mathbf{d})$ be an (M, Z)-CZCP. Then the cross Z-complementary ratio (CZCR) of $(\mathbf{c}, \mathbf{d})$ is defined as

$$\mathrm{CZCR} = \frac{Z}{Z_{max}},$$

where Z_{max} denotes the theoretical maximum ZCZ width.

For perfect CZCP (i.e., strengthened GCP) [2], $Z_{max} = M/2$, otherwise $Z_{max} = M/2 - 1$.

Lemma 3 ([6]): Let ($\mathbf{c}$, $\mathbf{d}$) be a binary sequence pair of length M. If

$$c_i = \frac{c_0}{d_0} d_i \text{ and } c_{M-1-i} = -\frac{c_0}{d_0} d_{M-1-i}, \ 0 \le i < Z.$$

Then the pair $(\mathbf{c}, \mathbf{d})$ satisfies

$$R_{\mathbf{c},\mathbf{d}}(\tau) + R_{\mathbf{d},\mathbf{c}}(\tau) = 0, \ M-Z \le |\tau| < M.$$

A method for extending the length of CZCPs by utilizing Turyn's construction has been proposed in [4].

Lemma 4: (Turyn's Construction, [4]) Let $(\mathbf{a}, \mathbf{b})$ be a binary GCP of length N, which is also an (N, Z_1)-CZCP, and $(\mathbf{c}, \mathbf{d})$ be a binary (M, Z)-CZCP. Then $(\mathbf{s}, \mathbf{t})$ is an (MN, ZN)-CZCP, where

$$\begin{aligned} \mathbf{s} &= \mathbf{c} \otimes (\mathbf{a} + \mathbf{b})/2 - \overleftarrow{\mathbf{d}} \otimes (\mathbf{b} - \mathbf{a})/2, \\ \mathbf{t} &= \mathbf{d} \otimes (\mathbf{a} + \mathbf{b})/2 + \overleftarrow{\mathbf{c}} \otimes (\mathbf{b} - \mathbf{a})/2. \end{aligned}$$

3. Construction of CZCPs from Turyn's Method

In this section, we will present a construction of CZCPs with large CZCR. Let's first consider the following lemma.

Lemma 5: Let $(\mathbf{c}, \mathbf{d})$ be a binary (M, Z)-CZCP. Then

1. If $c_Z = \frac{c_0}{d_0} d_Z$, we have

   $$R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(\tau) - R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(\tau) = 0, \text{ for any } M-1-Z \le |\tau| \le M-1.$$

2. If $c_{M-1-Z} = -\frac{c_0}{d_0} d_{M-1-Z}$, we have

   $$R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(\tau) - R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(\tau) = 0, \text{ for any } M-1-Z \le |\tau| \le M-1.$$

Proof: For $M-Z \le |\tau| \le M-1$, By Lemma 2, we have

$$R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(\tau) - R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(\tau) = 0.$$

For $|\tau| = M-1-Z$, If $c_Z = \frac{c_0}{d_0} d_Z$, we have

$$\begin{aligned} &R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(M-1-Z) \\ &= \sum_{i=0}^{Z} c_i \overleftarrow{c}_{(i+M-1-Z)} = \sum_{i=1}^{Z-1} c_i c_{Z-i} + 2c_0 c_Z \\ &= \sum_{i=1}^{Z-1} \frac{c_0}{d_0} d_i \frac{c_0}{d_0} d_{Z-i} + 2\frac{c_0}{d_0} d_0 \frac{c_0}{d_0} d_Z \\ &= \sum_{i=1}^{Z-1} d_i d_{Z-i} + 2d_0 d_Z = R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(M-1-Z). \end{aligned}$$

The third equality is due to the first point of Lemma 2.

Similarly, we can complete other proof.

Theorem 6: Let $\mathcal{S} = (\mathbf{a}, \mathbf{b})$ be a binary GCP of length N, which is also an (N, Z_1)-CZCP, and $\mathcal{T} = (\mathbf{c}, \mathbf{d})$ be a binary (M, Z)-CZCP. Let $(\mathbf{s}, \mathbf{t}) = \mathrm{Turyn}(\mathcal{S}, \mathcal{T})$, we have the following two conclusions:

1. If
$$\left(\frac{a_0}{b_0}+1\right)\left(c_Z-\frac{c_0}{d_0}d_Z\right) + \left(\frac{a_0}{b_0}-1\right)\left(c_{M-1-Z}+\frac{c_0}{d_0}d_{M-1-Z}\right)=0, \tag{1}$$
then $(\mathbf{s},\mathbf{t})$ is a binary $(MN, ZN+Z_1)$-CZCP.
2. Especially, if
$$c_Z=\frac{c_0}{d_0}d_Z \text{ and } c_{M-1-Z}=-\frac{c_0}{d_0}d_{M-1-Z}, \tag{2}$$
then $(\mathbf{s},\mathbf{t})$ is a binary $(MN,(Z+1)N-1)$-CZCP.

Proof: By Lemma 4, $(\mathbf{s},\mathbf{t})$ is an (MN, ZN)-CZCP.

For any time-shift $0\le\tau\le MN-1$, by the Euclidean division theorem, we have $\tau=k_1N+k_2$ where $0\le k_1\le M-1$, $0\le k_2\le N-1$.

Firstly, we calculate the sum of AACF of $\mathbf{s}$ and $\mathbf{t}$. By Definition 1, we have

$$\begin{aligned}&R_{\mathbf{s}}(\tau)+R_{\mathbf{t}}(\tau)\\ =&\frac{1}{2}\Big(R_{\mathbf{c}}(k_1)+R_{\mathbf{d}}(k_1)\Big)\Big(R_{\mathbf{a}}(k_2)+R_{\mathbf{b}}(k_2)\Big)+\frac{1}{2}\\ &\Big(R_{\mathbf{c}}(k_1+1)+R_{\mathbf{d}}(k_1+1)\Big)\Big(R_{\mathbf{a}}(N-k_2)+R_{\mathbf{b}}(N-k_2)\Big)\\ =&\frac{1}{2}\Big(R_{\mathbf{c}}(k_1)+R_{\mathbf{d}}(k_1)\Big)\Big(R_{\mathbf{a}}(k_2)+R_{\mathbf{b}}(k_2)\Big).\end{aligned} \tag{3}$$

The second equality is due to $(\mathbf{a},\mathbf{b})$ being a binary GCP.

Next, (3) will be discussed in following two cases:

- For $(N\le\tau<(Z+1)N)$ or $((M-Z)N\le\tau<MN)$, we have $(1\le k_1\le Z)$ or $(M-Z\le k_1\le M-1)$ and $0\le k_2\le N-1$. Since $(\mathbf{c},\mathbf{d})$ is an (M,Z)-CZCP, $R_{\mathbf{c}}(k_1)+R_{\mathbf{d}}(k_1)=0$. Hence
$$R_{\mathbf{s}}(\tau)+R_{\mathbf{t}}(\tau)=0. \tag{4}$$
- For $(1\le\tau<N)$ or $((M-1-Z)N<\tau<(M-Z)N)$, we have $(k_1=0$ or $k_1=M-1-Z)$ and $1\le k_2\le N-1$. Since $(\mathbf{a},\mathbf{b})$ is a GCP, $R_{\mathbf{a}}(k_2)+R_{\mathbf{b}}(k_2)=0$. Hence
$$R_{\mathbf{s}}(\tau)+R_{\mathbf{t}}(\tau)=0. \tag{5}$$

Summarizing equations (4) and (5), we can derive that

$$R_{\mathbf{s}}(\tau)+R_{\mathbf{t}}(\tau)=0,\ \tau\in\mathcal{T}_1\cup\mathcal{T}_2, \tag{6}$$

where $\mathcal{T}_1=\{1,2,\dots,(Z+1)N-1\}$ and $\mathcal{T}_2=\{(M-1-Z)N+1,\dots,MN-2,MN-1\}$.

Secondly, we calculate the sums of ACCF of $\mathbf{s}$ and $\mathbf{t}$. By Definition 1, we have

$$\begin{aligned}&R_{\mathbf{s},\mathbf{t}}(\tau)+R_{\mathbf{t},\mathbf{s}}(\tau)\\ =&\frac{R_{\mathbf{a}}(k_2)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_1)+R_{\mathbf{d},\mathbf{c}}(k_1)+R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1)+R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1)\\ &-R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1)-R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)-R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1)-R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big)\\ &+\frac{R_{\mathbf{b}}(k_2)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_1)+R_{\mathbf{d},\mathbf{c}}(k_1)-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1)-R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1)\\ &+R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1)+R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)-R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1)-R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big)\\ &+\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_1)+R_{\mathbf{d},\mathbf{c}}(k_1)-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1)+R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1)\\ &+R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1)-R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)+R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1)+R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big)\\ &+\frac{R_{\mathbf{b},\mathbf{a}}(k_2)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_1)+R_{\mathbf{d},\mathbf{c}}(k_1)+R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1)-R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1)\\ &-R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1)+R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)+R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1)+R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big)\\ &+\frac{R_{\mathbf{a}}(k_3)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_4)+R_{\mathbf{d},\mathbf{c}}(k_4)+R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_4)+R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_4)\\ &-R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_4)-R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_4)-R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_4)-R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_4)\Big)\\ &+\frac{R_{\mathbf{b}}(k_3)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_4)+R_{\mathbf{d},\mathbf{c}}(k_4)-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_4)-R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_4)\\ &+R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_4)+R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_4)-R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_4)-R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_4)\Big)\\ &+\frac{R_{\mathbf{a},\mathbf{b}}(k_3)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_4)+R_{\mathbf{d},\mathbf{c}}(k_4)+R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_4)-R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_4)\\ &-R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_4)+R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_4)+R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_4)+R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_4)\Big)\\ &+\frac{R_{\mathbf{b},\mathbf{a}}(k_3)}{4}\Big(R_{\mathbf{c},\mathbf{d}}(k_4)+R_{\mathbf{d},\mathbf{c}}(k_4)-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_4)+R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_4)\\ &+R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_4)-R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_4)+R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_4)+R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_4)\Big),\end{aligned} \tag{7}$$

where $k_3=N-k_2$, $k_4=k_1+1$.

For $(M-Z)N\le\tau<MN$, by Lemma 4, we have

$$R_{\mathbf{s},\mathbf{t}}(\tau)+R_{\mathbf{t},\mathbf{s}}(\tau)=0, \tag{8}$$

which holds due to $(\mathbf{s},\mathbf{t})$ being an (MN,ZN)-CZCP.

For $(M-1-Z)N<\tau<(M-Z)N$, then $k_1=M-1-Z$, $k_4=M-Z$ and $1\le k_2\le N-1$. Since $(\mathbf{c},\mathbf{d})$ is an (M,Z)-CZCP, we have $(\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}})$ is also an (M,Z)-CZCP according to Lemma 2. Thus,

$$R_{\mathbf{c},\mathbf{d}}(k_4)+R_{\mathbf{d},\mathbf{c}}(k_4)=0 \text{ and } R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_4)+R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_4)=0.$$
$$R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_4)-R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_4)=0 \text{ and } R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_4)-R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_4)=0.$$

By some elementary operations, we simplify (7) as

$$\begin{aligned}&R_{\mathbf{s},\mathbf{t}}(\tau)+R_{\mathbf{t},\mathbf{s}}(\tau)\\ =&\Big(\frac{R_{\mathbf{a}}(k_2)}{4}+\frac{R_{\mathbf{b}}(k_2)}{4}\Big)\\ &\Big(R_{\mathbf{c},\mathbf{d}}(k_1)+R_{\mathbf{d},\mathbf{c}}(k_1)-R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1)-R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big)\\ &+\Big(\frac{R_{\mathbf{a}}(k_2)}{4}-\frac{R_{\mathbf{b}}(k_2)}{4}\Big)\\ &\Big(R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1)+R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1)-R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1)-R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big)\\ &+\Big(\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{4}+\frac{R_{\mathbf{b},\mathbf{a}}(k_2)}{4}\Big)\end{aligned}$$

$$
\begin{aligned}
&\Big(R_{\mathbf{c},\mathbf{d}}(k_1) + R_{\mathbf{d},\mathbf{c}}(k_1) + R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big)\\
&+\Big(\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{4} - \frac{R_{\mathbf{b},\mathbf{a}}(k_2)}{4}\Big)\\
&\Big(-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) + R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big).
\end{aligned}
$$

Since $(\mathbf{a},\mathbf{b})$ is a binary GCP, we have $R_{\mathbf{a}}(k_2) + R_{\mathbf{b}}(k_2) = 0$. Hence, we can obtain

$$
\begin{aligned}
&R_{\mathbf{s},\mathbf{t}}(\tau) + R_{\mathbf{t},\mathbf{s}}(\tau)\\
=&\Big(\frac{R_{\mathbf{a}}(k_2)}{4} - \frac{R_{\mathbf{b}}(k_2)}{4}\Big)\\
&\Big(R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) - R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big)\\
&+\Big(\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{4} + \frac{R_{\mathbf{b},\mathbf{a}}(k_2)}{4}\Big)\\
&\Big(R_{\mathbf{c},\mathbf{d}}(k_1) + R_{\mathbf{d},\mathbf{c}}(k_1) + R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big)\\
&+\Big(\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{4} - \frac{R_{\mathbf{b},\mathbf{a}}(k_2)}{4}\Big)\\
&\Big(-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) + R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big).
\end{aligned}
\tag{9}
$$

Next, (9) will be discussed in following two cases:

1) If (1) is satisfied, we can obtain ($c_Z = \frac{c_0}{d_0}d_Z$ and $\frac{a_0}{b_0} = 1$) or ($c_{M-1-Z} = -\frac{c_0}{d_0}d_{M-1-Z}$ and $\frac{a_0}{b_0} = -1$). Due to similarity, we only consider $c_Z = \frac{c_0}{d_0}d_Z$ and $\frac{a_0}{b_0} = 1$ here.

We consider $(M-Z)N - Z_1 \le \tau \le (M-Z)N - 1$, then $k_1 = M-1-Z$ and $N - Z_1 \le k_2 \le N-1$.

$$
\begin{aligned}
&R_{\mathbf{s},\mathbf{t}}(\tau) + R_{\mathbf{t},\mathbf{s}}(\tau)\\
=&\Big(\frac{R_{\mathbf{a}}(k_2)}{4} - \frac{R_{\mathbf{b}}(k_2)}{4}\Big)\\
&\Big(R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) - R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big)\\
&+\Big(\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{4} - \frac{R_{\mathbf{b},\mathbf{a}}(k_2)}{4}\Big)\\
&\Big(-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) + R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big)\\
=&\frac{R_{\mathbf{a}}(k_2)}{2}\Big(R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) - R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big)\\
&+\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{2}\\
&\Big(-R_{\mathbf{d},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) + R_{\mathbf{c},\overleftarrow{\mathbf{c}}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big)\\
=&\frac{1}{2}\Big(R_{\mathbf{a}}(k_2) + R_{\mathbf{a},\mathbf{b}}(k_2)\Big)\Big(R_{\overleftarrow{\mathbf{d}},\mathbf{d}}(k_1) - R_{\overleftarrow{\mathbf{c}},\mathbf{c}}(k_1)\Big).
\end{aligned}
$$

The first and second equalities stem from $(\mathbf{a},\mathbf{b})$ being an (N, Z_1)-CZCP, and the third equality arises from the first point of Lemma 5.

By Lemma 2, we also have $a_i = b_i, a_{N-1-i} = -b_{N-1-i}$ for any $0 \le i \le Z_1 - 1$. Then, for any $N - Z_1 \le k_2 \le N-1$,

$$
\begin{aligned}
R_{\mathbf{a}}(k_2) + R_{\mathbf{a},\mathbf{b}}(k_2) &= \sum_{i=0}^{N-1-k_2} a_i a_{i+k_2} + \sum_{i=0}^{N-1-k_2} a_i b_{i+k_2}\\
&= 0.
\end{aligned}
$$

Hence, we have

$$
R_{\mathbf{s},\mathbf{t}}(\tau) + R_{\mathbf{t},\mathbf{s}}(\tau) = 0. \tag{10}
$$

From the equations (6), (8) and (10), we can derive that when (1) holds, $(\mathbf{s},\mathbf{t})$ is a binary $(MN, ZN + Z_1)$-CZCP.

2) If (2) is satisfied, we consider $(M-1-Z)N + 1 \le \tau \le (M-Z)N - 1$, then $k_1 = M-1-Z$ and $1 \le k_2 \le N-1$.

From Lemma 5, we have

$$
\begin{aligned}
&R_{\mathbf{s},\mathbf{t}}(\tau) + R_{\mathbf{t},\mathbf{s}}(\tau)\\
=&\Big(\frac{R_{\mathbf{a},\mathbf{b}}(k_2)}{4} + \frac{R_{\mathbf{b},\mathbf{a}}(k_2)}{4}\Big)\\
&\Big(R_{\mathbf{c},\mathbf{d}}(k_1) + R_{\mathbf{d},\mathbf{c}}(k_1) + R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1)\Big).
\end{aligned}
$$

Since $(\mathbf{c},\mathbf{d})$ is a binary (M, Z)-CZCP, by Lemma 2, we can obtain that for any $i \in \{0, 1, \cdots, Z\}$

$$
c_i = \frac{c_0}{d_0}d_i \text{ and } c_{M-1-i} = -\frac{c_0}{d_0}d_{M-1-i}.
$$

By Lemma 3, we have for $k_1 = M-1-Z$, $R_{\mathbf{c},\mathbf{d}}(k_1) + R_{\mathbf{d},\mathbf{c}}(k_1) = 0$, $R_{\overleftarrow{\mathbf{c}},\overleftarrow{\mathbf{d}}}(k_1) + R_{\overleftarrow{\mathbf{d}},\overleftarrow{\mathbf{c}}}(k_1) = 0$. Then, we have

$$
R_{\mathbf{s},\mathbf{t}}(\tau) + R_{\mathbf{t},\mathbf{s}}(\tau) = 0. \tag{11}
$$

From the equations (6), (8) and (11), we can derive that when (2) holds, $(\mathbf{s},\mathbf{t})$ is a binary $(MN, (Z+1)N - 1)$-CZCP.

Remark 7: (i) We can extend any (M, Z)-CZCP with properties (1) or properties (2) to $(MN, ZN + Z_1)$ or $(MN, (Z+1)N - 1)$-CZCPs by Theorem 6, respectively.

(ii) The CZCPs we used with parameters (18,7), (22,9), (34,14), (38,14) are presented in Table 1.

Table 1 CZCPs of Lengths $M \in \{18, 22, 34, 38\}$

K_M	(M, Z)	$\begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix}$
K_{18}	$(18, 7)$	$\begin{pmatrix}+++-++--+----+-+--\\ +++-++----+++-+-++\end{pmatrix}$
K_{22}	$(22, 9)$	$\begin{pmatrix}++++-+-+++-+----+--++-\\ ++++-+-++---++++-++--+\end{pmatrix}$
K_{34}	$(34, 14)$	$\begin{pmatrix}\begin{bmatrix}-+++-+-----+---++\\ -+-++-----++---+-\end{bmatrix}\\ \begin{bmatrix}-+++-+-----+--+-+\\ --+--+++++--+++-+\end{bmatrix}\end{pmatrix}$
K_{38}	$(38, 14)$	$\begin{pmatrix}\begin{bmatrix}-+----+++---+--++-+\\ +++-+-----+---+-++-\end{bmatrix}\\ \begin{bmatrix}-+----+++---+--+++-\\ ++-+-+++++-+++-+--+\end{bmatrix}\end{pmatrix}$

4. Conclusion

In this paper, we introduced a novel framework for constructing binary CZCPs. By exploring the properties of these seed CZCPs, we obtained CZCPs with large CZCR, which has not been reported before. In future work, we will continue to investigate CZCPs satisfying properties (1) or (2).

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