IEICE TRANSACTIONS

on Fundamentals of Electronics, Communications and Computer Sciences

DOI:10.1587/transfun.2024TAP0006

Publicized:2024/08/27

This advance publication article will be replaced by the finalized version after proofreading.

A PUBLICATION OF THE ENGINEERING SCIENCES SOCIETY

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PAPER Special Section on Information Theory and Its Applications

Mean Squared Error Analysis of Noisy Average Consensus

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SUMMARY A continuous-time average consensus system is a linear dynamical system defined over a graph, where each node has its own state value that evolves according to a simultaneous linear differential equation. A node is allowed to interact with neighboring nodes. Average consensus is a phenomenon that the all the state values converge to the average of the initial state values. In this paper, we assume that a node can communicate with neighboring nodes through an additive white Gaussian noise channel. We first formulate the noisy average consensus system by using a stochastic differential equation (SDE), which allows us to use the Euler-Maruyama method, a numerical technique for solving SDEs. By studying the stochastic behavior of the residual error of the Euler-Maruyama method, we arrive at the covariance evolution equation. The analysis of the residual error leads to a compact formula for mean squared error (MSE), which shows that the sum of the inverse eigenvalues of the Laplacian matrix is the most dominant factor influencing the MSE.
key words: *average consensus, stochastic differential equation, Euler-Maruyama method, MSE*

1. Introduction

Continuous-time *average consensus system* is a linear dynamical system defined over a connected graph [1]. Each node has its own state value, and it evolves according to a simultaneous linear differential equation where a node is only allowed to interact with neighboring nodes. The ordinary differential equation (ODE) at the node $i (1 \le i \le n)$ governing the evolution of the state value $x_i(t)$ of the node i is given by

$$\frac{dx_i(t)}{dt} = -\sum_{j \in \mathcal{N}_i} \mu_{ij}(x_i(t) - x_j(t)). \quad (1)$$

The set $\mathcal{N}_i$ denote the neighboring nodes of node i, while the positive scalar μ_{ij} denotes the edge weight associated with the edge (i, j). The same ODE applies to all other nodes as well. These dynamics gradually decrease the differences between the state values of neighboring nodes, leading to a phenomenon called average consensus that the all the state values converge to the average of the initial state values [2].

The average consensus system has been studied in numerous fields such as multi-agent control [3], distributed algorithm [4], formation control [5]. An excellent survey on average consensus systems can be found in [1].

In this paper, we will examine average consensus systems within the context of communications across noisy channels, such as wireless networks. Specifically, we consider the scenario in which nodes engage in local wireless communication, such as drones flying in the air or sensors dispersed across a designated area. It is assumed that each node can only communicate with neighboring nodes via an additive white Gaussian noise (AWGN) channel. The objective of the communication is to aggregate the information held by all nodes through the application of average consensus systems. As previously stated, the consensus value is the average of the initial state values.

In this setting, we must account for the impact of Gaussian noise on the differential equations. The differential equation for a *noisy average consensus system* takes the form:

$$\frac{dx_i(t)}{dt} = -\sum_{j \in \mathcal{N}_i} \mu_{ij}(x_i(t) - x_j(t)) + \alpha W_i(t), \quad (2)$$

where $W_i(t)$ represents an additive white Gaussian process, and α is a positive constant. The noise $W_i(t)$ can be considered as the sum of the noises occurring on the edges adjacent to the node i. In a noiseless average consensus system, it is well-established that the second smallest eigenvalue of the Laplacian matrix of the graph determines the convergence speed to the average [4]. The convergence behavior of a noisy system may be quite different from that of the noiseless system due to the presence of edge noise. However, the stochastic dynamics of such a system has not yet been studied. Studies on discrete-time consensus protocols subject to additive noise can be found in [6][7], but to the best of our knowledge, there are no prior studies on continuous-time noisy consensus systems.

The main goal of this paper is to study the stochastic dynamics of continuous-time noisy average consensus system. The theoretical understanding of the stochastic behavior of such systems will be valuable for various areas such as multi-agent control and the design of consensus-based distributed algorithms for noisy environments.

The primary contributions of this paper are as follows. We first formulate the noisy average consensus systems using stochastic differential equations (SDE) [8][9]. This SDE formulation facilitates mathematically rigorous treatment of noisy average consensus. We use the Euler-Maruyama method [8] for solving the SDE, which is a numerical method for solving SDEs. We derive a closed-form mean squared error (MSE) formula by analyzing the stochastic behavior of the residual errors in the Euler-Maruyama method. We show that the MSE is dominated by the sum of the inverse

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eigenvalues of the Laplacian matrix.

The outline of the paper is as follows. In Section 2, we introduce the mathematical notation used throughout the paper, and then provide the definition and fundamental properties of average consensus systems. In Section 3, we define a noisy average consensus system as a SDE. In Section 4, we present an analysis of the stochastic behavior of the consensus error and derive a concise MSE formula. Finally, in Section 5, we conclude the discussion.

2. Preliminaries

2.1 Notation

The following notation will be used throughout this paper. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ represent the set of real numbers and the set of positive real numbers, respectively. The one dimensional Gaussian distribution with mean μ and variance σ^2 is denoted by $\mathcal{N}(\mu, \sigma^2)$. The multivariate Gaussian distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is represented by $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The expectation operator is denoted by $\mathsf{E}[\cdot]$. The notation $\mathrm{diag}(\boldsymbol{x})$ is the diagonal matrix whose diagonal elements are given by $\boldsymbol{x} \in \mathbb{R}^n$. The matrix exponential $\exp(\boldsymbol{X})(\boldsymbol{X} \in \mathbb{R}^{n\times n})$ is defined by

$$\exp(\boldsymbol{X}) \equiv \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{X}^k. \tag{3}$$

The Frobenius norm of $\boldsymbol{X} \in \mathbb{R}^{n\times n}$ is denoted by $\|\boldsymbol{X}\|_F$. The notation $[n]$ denotes the set of consecutive integers from 1 to n.

2.2 Average Consensus

Let $G \equiv (V, E)$ be a connected undirected graph where $V = [n]$. Suppose that a node $i \in V$ can be regarded as an *agent* communicating over the graph G. Namely, a node i and a node j can communicate with each other if $(i, j) \in E$. We will not distinguish (i, j) and (j, i) because the graph G is undirected.

Each node i has a state value $x_i(t) \in \mathbb{R}(t \geq 0)$ where $t \in \mathbb{R}$ represents continuous-time variable. The neighborhood of a node $i \in V$ is represented by

$$\mathcal{N}_i \equiv \{j \in V : (j, i) \in E, i \neq j\}. \tag{4}$$

Note that the node i is excluded from $\mathcal{N}_i$. For any time t, a node $i \in V$ can access the self-state $x_i(t)$ and the state values of its neighborhood, i.e., $x_j(t), j \in \mathcal{N}_i$ but cannot access to the other state values.

In this section, we briefly review the basic properties of the average consensus processes [1]. We now assume that the set of state values

$$\boldsymbol{x}(t) \equiv (x_1(t), x_2(t), \ldots, x_n(t))^T \tag{5}$$

are evolved according to the simultaneous differential equations

$$\frac{dx_i(t)}{dt} = -\sum_{j\in\mathcal{N}_i} \mu_{ij}(x_i(t) - x_j(t)), \quad i \in [n], \tag{6}$$

where the initial condition is

$$\boldsymbol{x}(0) = \boldsymbol{c} \equiv (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n. \tag{7}$$

The edge weight $\mu_{ij} \in \mathbb{R}(\mu_{ij} > 0)$ follows the symmetric condition

$$\mu_{ij} = \mu_{ji}, \quad i \in [n], j \in [n]. \tag{8}$$

Let $\boldsymbol{\Delta} \equiv (\Delta_1, \Delta_2, \ldots, \Delta_n)^T \in \mathbb{R}_+^n$ be a *degree sequence* where Δ_i is defined by

$$\Delta_i \equiv \sum_{j\in\mathcal{N}_i} \mu_{ij}, \quad i \in [n]. \tag{9}$$

The continuous-time dynamical system (6) is called an *average consensus system* because a state value converges to the average of the initial state values at the limit of $t \to \infty$, i.e,

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \frac{1}{n}\left(\sum_{i=1}^{n} c_i\right)\mathbf{1} = \gamma\mathbf{1}, \tag{10}$$

where the vector $\mathbf{1}$ represents $(1, 1, \ldots, 1)^T$ and γ is defined by

$$\gamma \equiv \frac{1}{n}\sum_{i=1}^{n} c_i. \tag{11}$$

We define the Laplacian matrix $\boldsymbol{L} \equiv \{L_{ij}\} \in \mathbb{R}^{n\times n}$ of this consensus system as follows:

$$L_{ij} = \Delta_i, \quad i = j,\ i \in [n], \tag{12}$$

$$L_{ij} = -\mu_{ij}, \quad i \neq j \text{ and } (i, j) \in E, \tag{13}$$

$$L_{ij} = 0, \quad i \neq j \text{ and } (i, j) \notin E. \tag{14}$$

From this definition, a Laplacian matrix satisfies

$$\boldsymbol{L}\mathbf{1} = \mathbf{0}, \tag{15}$$

$$\mathrm{diag}(\boldsymbol{L}) = \boldsymbol{\Delta}, \tag{16}$$

$$\boldsymbol{L} = \boldsymbol{L}^T. \tag{17}$$

Note that the eigenvalues of the Laplacian matrix $\boldsymbol{L}$ are nonnegative real because $\boldsymbol{L}$ is a positive semi-definite symmetric matrix. Let $\lambda_1 = 0 < \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $\boldsymbol{L}$ and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_n$ be the corresponding orthonormal eigenvectors. The first eigenvector $\boldsymbol{\xi}_1 \equiv (1/\sqrt{n})\mathbf{1}$ is corresponding to the eigenvalue $\lambda_1 = 0$, which results in $\boldsymbol{L}\boldsymbol{\xi}_1 = 0$.

By using the notion of the Laplacian matrix, the dynamical system (6) can be compactly rewritten as

$$\frac{d\boldsymbol{x}(t)}{dt} = -\boldsymbol{L}\boldsymbol{x}(t), \tag{18}$$

where the initial condition is $\boldsymbol{x}(0) = \boldsymbol{c}$. The dynamical behaviors of the average consensus system (18) are thus characterized by the Laplacian matrix $\boldsymbol{L}$. Since the ODE (18) is a linear ODE, it can be easily solved. The solution of the ODE (18) is given by

$$\boldsymbol{x}(t) = \exp(-\boldsymbol{L}t)\boldsymbol{x}(0), \quad t \geq 0. \tag{19}$$

Let $\boldsymbol{U} \equiv (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_n) \in \mathbb{R}^{n\times n}$ where $\boldsymbol{U}$ is an orthogonal matrix. The Laplacian matrix $\boldsymbol{L}$ can be diagonalized by using $\boldsymbol{U}$, i.e.,

$$\boldsymbol{L} = \boldsymbol{U}\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\boldsymbol{U}^T. \tag{20}$$

On the basis of the diagonalization, we have the spectral expansion of the matrix exponential:

$$\begin{aligned}\exp(-\boldsymbol{L}t) &= \exp(-\boldsymbol{U}\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\boldsymbol{U}^T t)\\ &= \boldsymbol{U}\exp(-\mathrm{diag}(\lambda_1, \ldots, \lambda_n)t)\boldsymbol{U}^T\\ &= \sum_{i=1}^{n} \exp(-\lambda_i t)\boldsymbol{\xi}_i\boldsymbol{\xi}_i^T. \end{aligned}\tag{21}$$

Substituting this to $\boldsymbol{x}(t) = \exp(-\boldsymbol{L}t)\boldsymbol{x}(0)$, we immediately have

$$\boldsymbol{x}(t) = \frac{1}{n}\mathbf{1}(\mathbf{1}^T)\boldsymbol{c} + \sum_{i=2}^{n} \exp(-\lambda_i t)\boldsymbol{\xi}_i\boldsymbol{\xi}_i^T\boldsymbol{c}. \tag{22}$$

The second term of the right-hand side converges to zero since $\lambda_k > 0$ for $k = 2, 3, \ldots, n$. This explains why average consensus happens, i.e., the convergence to the average of the initial state values (10). The second smallest eigenvalue λ_2, called *algebraic connectivity* [11], determines the convergence speed because $\exp(-\lambda_2 t)\boldsymbol{\xi}_2\boldsymbol{\xi}_2^{\mathsf{T}}$ shows the slowest convergence in the second term.

3. Noisy average consensus system

3.1 SDE formulation

The dynamical model (2) containing a white Gaussian noise process is mathematically challenging to handle. We will use a common approach [8] of approximating the white Gaussian process by using the standard Wiener process. Instead of model (2), we will focus on the following stochastic differential equation (SDE) [9]

$$d\boldsymbol{x}(t) = -\boldsymbol{L}\boldsymbol{x}(t)dt + \alpha d\boldsymbol{b}(t) \tag{23}$$

to study the noisy average consensus system. The parameter α is a positive real number, and it represents the *intensity of the noises*. The stochastic term $\boldsymbol{b}(t)$ represents the n-dimensional standard Wiener process. The elements of $\boldsymbol{b}(t) = (b_1(t), b_2(t), \ldots, b_n(t))^T$ are independent one dimensional-standard Wiener processes. For the Wiener process $b(t)$, we have $b(0) = 0$, $E[b(t)] = 0$, and it satisfies

$$b(t) - b(s) \sim \mathcal{N}(0, t-s), \ 0 \leq s \leq t. \tag{24}$$

3.2 Approaches for studying stochastic dynamics

Our primary objective in the following analysis is to investigate the stochastic dynamics of the noisy average consensus system, focusing on deriving the mean and covariance of the solution $\boldsymbol{x}(t)$ for the SDE (23) because the solution of the SDE is a stochastic process.

There are two approaches to analyze the system. The first approach relies on the established theory of *Ito calculus* [9], which is used to handle stochastic integrals directly (see Fig. 1). Ito calculus can be applied to derive the first and second moments of the solution of (23).

Alternatively, the second approach employs the Euler-Maruyama (EM) method [8] and utilizes the weak convergence property [8] of the EM method. We will adopt the latter approach in our analysis, as it does not require knowledge of advanced stochastic calculus if we accept the weak convergence property. Additionally, this approach can be naturally extended to the analysis on the discrete-time noisy average consensus system. Namely, the analysis presented in Section 4 is essentially an analysis for a discrete-time noisy consensus system.

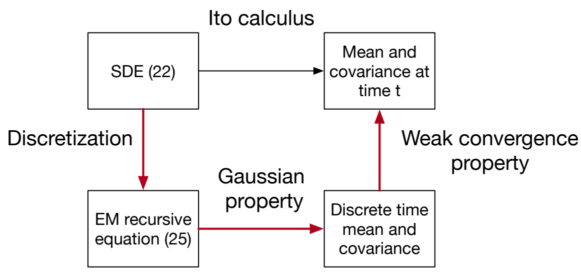

Fig. 1 Two approaches for deriving the mean and covariance of $\boldsymbol{x}(t)$. This paper follows the lower path using the EM method.

3.3 Euler-Maruyama method

We use the *Euler-Maruyama method* corresponding to this SDE so as to study the stochastic behavior of the solution of the SDE (23) defined above. The EM method is well-known numerical method for solving SDEs [8].

Assume that we need numerical solutions of a SDE in the time interval $0 \leq t \leq T$. We divide this interval into N bins and let $t_k \equiv k\eta$, $k = 0, 1, \ldots, N$ where the interval η is given by $\eta \equiv T/N$. Let us define a discretized sample $\boldsymbol{x}^{(k)}$ be $\boldsymbol{x}^{(k)} \equiv \boldsymbol{x}(t_k)$. It should be noted that, the choice of the width η is crucial in order to ensure the stability and the accuracy of the EM method. A small width leads to a more accurate solution, but requires more computational time. A large width may be computationally efficient but may lead to instability in the solution.

The recursive equation of the EM method corresponding to SDE (23) is given by

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \eta \boldsymbol{L}\boldsymbol{x}^{(k)} + \alpha \boldsymbol{w}^{(k)},\ k = 0, 1, 2, \dots, N-1, \quad (25)$$

where each element of $\boldsymbol{w}^{(k)} \equiv (w_1^{(k)}, w_2^{(k)}, \dots, w_n^{(k)})^T$ follows $\boldsymbol{w}^{(k)} \sim \mathcal{N}(0, \eta \boldsymbol{I})$. In the following discussion, we will use the equivalent expression [8]:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \eta \boldsymbol{L}\boldsymbol{x}^{(k)} + \alpha\sqrt{\eta}\boldsymbol{z}^{(k)},\ k = 0, 1, 2, \dots, N-1, \quad (26)$$

where $\boldsymbol{z}^{(k)}$ is a random vector following the multivariate Gaussian distribution $\mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$. The initial vector $\boldsymbol{x}^{(0)}$ is set to be $\boldsymbol{c}$. This recursive equation will be referred to as the *Euler-Maruyama recursive equation*.

Figure 2 presents a solution evaluated with the EM method. The cycle graph with 10 nodes with the degree sequence $\boldsymbol{\Delta} = (2, 2, \dots, 2)$ and edge weight 1 is assumed. The initial value is randomly initialized as $\boldsymbol{x}(0) \sim \mathcal{N}(0, \boldsymbol{I})$. We can confirm that the state values are certainly converging to the average value γ in the case of noiseless case (left). On the other hand, the state vector fluctuates around the average in the noisy case (right).

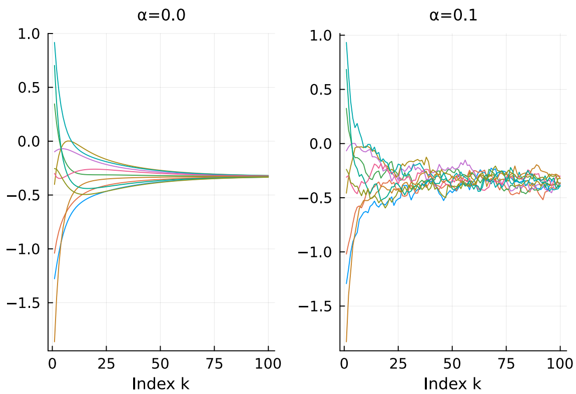

Fig. 2 Trajectories of $\boldsymbol{x}(t_k) = (x_1(t_k), \dots, x_n(t_k))$ estimated by using the EM method. Cycle graph with 10 nodes were used. The range [0, 10.0] are discretized with $N = 100$ points. The consensus average value is $\gamma = -0.3267$. Left panel: noiseless case ($\alpha = 0.0$), Right panel: noisy case ($\alpha = 0.1$).

4. Analysis for Noisy average consensus

4.1 Recursive equation for residual error

In the following, we will analyze the stochastic behavior of the residual error. This will be the basis for the MSE formula to be presented.

Recall that the initial state vector is $\boldsymbol{c} = (c_1, c_2, \dots, c_n)^T$ and that the average of the initial values is denoted by γ. Since the set of eigenvectors $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n\}$ of $\boldsymbol{L}$ is an orthonormal base, we can expand the initial state vector $\boldsymbol{c}$ as

$$\boldsymbol{c} = \zeta_1\boldsymbol{\xi}_1 + \zeta_2\boldsymbol{\xi}_2 + \dots + \zeta_n\boldsymbol{\xi}_n, \quad (27)$$

where the coefficient is obtained by $\zeta_i = \boldsymbol{c}^T\boldsymbol{\xi}_i (i \in [n])$. Note that $\zeta_1\boldsymbol{\xi}_1 = \gamma\boldsymbol{1}$ holds.

At the initial index $k = 0$, the Euler-Maruyama recursive equation becomes

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} - \eta\boldsymbol{L}\boldsymbol{x}^{(0)} + \alpha\sqrt{\eta}\boldsymbol{z}^{(0)}. \quad (28)$$

Since $\boldsymbol{L}\boldsymbol{\xi}_1 = \boldsymbol{0}$, we have

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} - \eta\boldsymbol{L}(\boldsymbol{x}^{(0)} - \gamma\boldsymbol{1}) + \alpha\sqrt{\eta}\boldsymbol{z}^{(0)},$$

Subtracting $\gamma\boldsymbol{1}$ from the both sides, we get

$$\boldsymbol{x}^{(1)} - \gamma\boldsymbol{1} = (\boldsymbol{I} - \eta\boldsymbol{L})(\boldsymbol{x}^{(0)} - \gamma\boldsymbol{1}) + \alpha\sqrt{\eta}\boldsymbol{z}^{(0)}. \quad (29)$$

For the index $k \ge 1$, the Euler-Maruyama recursive equation from (26) can be written as

$$\boldsymbol{x}^{(k+1)} = (\boldsymbol{I} - \eta\boldsymbol{L})\boldsymbol{x}^{(k)} + \alpha\sqrt{\eta}\boldsymbol{z}^{(k)}. \quad (30)$$

Subtracting $\gamma\boldsymbol{1}$ from the both sides, we have

$$\boldsymbol{x}^{(k+1)} - \gamma\boldsymbol{1} = (\boldsymbol{I} - \eta\boldsymbol{L})\boldsymbol{x}^{(k)} - \gamma\boldsymbol{1} + \alpha\sqrt{\eta}\boldsymbol{z}^{(k)}. \quad (31)$$

By using the relation $(\boldsymbol{I} - \eta\boldsymbol{L})\gamma\boldsymbol{1} = \gamma\boldsymbol{1}$, we can rewrite the above equation as

$$\begin{aligned}\boldsymbol{x}^{(k+1)} - \gamma\boldsymbol{1} &= (\boldsymbol{I} - \eta\boldsymbol{L})\boldsymbol{x}^{(k)} - (\boldsymbol{I} - \eta\boldsymbol{L})\gamma\boldsymbol{1} + \alpha\sqrt{\eta}\boldsymbol{z}^{(k)} \\ &= (\boldsymbol{I} - \eta\boldsymbol{L})(\boldsymbol{x}^{(k)} - \gamma\boldsymbol{1}) + \alpha\sqrt{\eta}\boldsymbol{z}^{(k)}. \quad (32)\end{aligned}$$

It can be confirmed the above recursion (32) is consistent with the initial equation (29). We here summarize the above argument as the following lemma.

Lemma 1: Let $\boldsymbol{e}^{(k)} \equiv \boldsymbol{x}^{(k)} - \gamma\boldsymbol{1}$ be the *residual error* at index k. The evolution of the residual error of the EM method is described by

$$\boldsymbol{e}^{(k+1)} = (\boldsymbol{I} - \eta\boldsymbol{L})\boldsymbol{e}^{(k)} + \alpha\sqrt{\eta}\boldsymbol{z}^{(k)} \quad (33)$$

for $k \ge 0$.

The residual error $\boldsymbol{e}^{(k)}$ denotes the error between the average vector $\gamma\boldsymbol{1}$ and the state vector $\boldsymbol{x}^{(k)}$ at time index k. By analyzing the statistical behavior of $\boldsymbol{e}^{(k)}$, we can gain insight into the stochastic properties of the dynamics of the noisy consensus system.

4.2 Asymptotic mean of residual error

Let a vector $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Recall that the vector obtained by a linear map $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ also follows the Gaussian distribution, i.e.,

$$\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{A}\boldsymbol{\mu}, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T), \quad (34)$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. If two Gaussian vectors $\boldsymbol{a} \sim \mathcal{N}(\boldsymbol{\mu}_a, \boldsymbol{\Sigma}_a)$ and $\boldsymbol{b} \sim \mathcal{N}(\boldsymbol{\mu}_b, \boldsymbol{\Sigma}_b)$ are independent, the sum $\boldsymbol{z} = \boldsymbol{a} + \boldsymbol{b}$ becomes also Gaussian, i.e,

$$\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{\mu}_a + \boldsymbol{\mu}_b, \boldsymbol{\Sigma}_a + \boldsymbol{\Sigma}_b). \quad (35)$$

In the recursive equation (33), it is evident that $\boldsymbol{e}^{(1)}$ follows a multivariate Gaussian distribution because

$$\boldsymbol{e}^{(1)} = (\boldsymbol{I} - \eta \boldsymbol{L})(\boldsymbol{c} - \gamma \boldsymbol{1}) + \alpha\sqrt{\eta}\boldsymbol{z}^{(0)} \tag{36}$$

is the sum of a constant vector and a Gaussian random vector. From the above properties of Gaussian random vectors, the residual error vector $\boldsymbol{e}^{(k)}$ follows the multivariate Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}^{(k)}, \boldsymbol{\Sigma}^{(k)})$ where the mean vector $\boldsymbol{\mu}^{(k)}$ and the covariance matrix $\boldsymbol{\Sigma}^{(k)}$ are recursively determined by

$$\boldsymbol{\mu}^{(k+1)} = (\boldsymbol{I} - \eta \boldsymbol{L})\boldsymbol{\mu}^{(k)}, \tag{37}$$

$$\boldsymbol{\Sigma}^{(k+1)} = (\boldsymbol{I} - \eta \boldsymbol{L})\boldsymbol{\Sigma}^{(k)}(\boldsymbol{I} - \eta \boldsymbol{L})^T + \alpha^2\eta \boldsymbol{I} \tag{38}$$

for $k \ge 0$ where the initial values are formally given by

$$\boldsymbol{\mu}^{(0)} = \boldsymbol{c} - \gamma\boldsymbol{1}, \tag{39}$$

$$\boldsymbol{\Sigma}^{(0)} = \boldsymbol{O}. \tag{40}$$

Solving the recursive equation, we can get the asymptotic mean formula as follows.

Lemma 2: Suppose that $T > 0$ is given. The asymptotic mean of residual error as $N \to \infty$ is given by

$$\lim_{N\to\infty} \boldsymbol{\mu}^{(N)} = \exp(-\boldsymbol{L}T)(\boldsymbol{c} - \gamma\boldsymbol{1}). \tag{41}$$

(Proof) We formally set the initial mean as $\boldsymbol{\mu}^{(0)} = \boldsymbol{c} - \gamma\boldsymbol{1}$. The mean recursion is given as $\boldsymbol{\mu}^{(k)} = (\boldsymbol{I} - \eta\boldsymbol{L})^k(\boldsymbol{c} - \gamma\boldsymbol{1})$ for $k \ge 1$. Recall that the eigenvalue decomposition of $\boldsymbol{L}$ is given by $\boldsymbol{L} = \boldsymbol{U}\mathrm{diag}(\lambda_1, \dots, \lambda_n)\boldsymbol{U}^T$. From

$$\boldsymbol{I} - \eta\boldsymbol{L} = \boldsymbol{U}(\boldsymbol{I} - \eta\mathrm{diag}(\lambda_1, \dots, \lambda_n))\boldsymbol{U}^T, \tag{42}$$

we have

$$(\boldsymbol{I} - \eta\boldsymbol{L})^k = \boldsymbol{U}\mathrm{diag}((1-\eta\lambda_1)^k, \dots, (1-\eta\lambda_n)^k)\boldsymbol{U}^T. \tag{43}$$

This implies, from the definition of exponential function,

$$\lim_{N\to\infty}\left(\boldsymbol{I} - \frac{T}{N}\boldsymbol{L}\right)^N = \exp(-\boldsymbol{L}T), \tag{44}$$

where $\eta = T/N$. □

It is easy to confirm that the claim of this lemma is consistent with the continuous solution of noiseless case (19). Namely, at the limit of $\alpha \to 0$, the state evolution of the noisy system converges to that of the noiseless system.

4.3 Asymptotic covariance of residual error

We here discuss the asymptotic behavior of the covariance matrix $\boldsymbol{\Sigma}^{(N)}$ at the limit of $N \to \infty$.

Lemma 3: Suppose that $T > 0$ is given. The asymptotic covariance matrix at $N \to \infty$ is given by

$$\lim_{N\to\infty} \boldsymbol{\Sigma}^{(N)} = \boldsymbol{U}\mathrm{diag}\left(\alpha^2 T, \theta_2, \theta_3, \dots, \theta_n\right)\boldsymbol{U}^T, \tag{45}$$

where θ_i is defined by

$$\theta_i \equiv \frac{\alpha^2}{2\lambda_i}\left(1 - e^{-2\lambda_i T}\right). \tag{46}$$

(Proof) Recall that

$$\boldsymbol{I} - \eta\boldsymbol{L} = \boldsymbol{U}\mathrm{diag}(1, 1-\eta\lambda_2 \dots, 1-\eta\lambda_n)\boldsymbol{U}^T. \tag{47}$$

Let $\boldsymbol{\Sigma}^{(k)} = \boldsymbol{U}\mathrm{diag}(s_1^{(k)}, \dots, s_n^{(k)})\boldsymbol{U}^T$. A spectral representation of the covariance evolution (38) is thus given by

$$\mathrm{diag}(s_1^{(k+1)}, \dots, s_n^{(k+1)}) = \mathrm{diag}(s_1^{(k)}, s_2^{(k)}(1-\eta\lambda_2)^2 \dots, s_n^{(k)}(1-\eta\lambda_n)^2) + \alpha^2\eta\boldsymbol{I}, \tag{48}$$

where $s_i^{(0)} = 0$ for $i = 1, 2, \dots, n$. The first component follows a recursion $s_1^{(k+1)} = s_1^{(k)} + \alpha^2\eta$ and thus we have $s_1^{(N)} = \alpha^2\eta N = \alpha^2 T$. Another component follows

$$s_i^{(k+1)} = s_i^{(k)}(1-\eta\lambda_i)^2 + \alpha^2\eta. \tag{49}$$

Let us consider the characteristic equation of (49) which is given by

$$s = s(1-\eta\lambda_i)^2 + \alpha^2\eta. \tag{50}$$

The solution of the equation is given by

$$s = \frac{\alpha^2\eta}{1-(1-\eta\lambda_i)^2}. \tag{51}$$

The above recursive equation (49) thus can be transformed as

$$s_i^{(k+1)} - s = (s_i^{(k)} - s)(1-\eta\lambda_i)^2. \tag{52}$$

From the above equation, $s_i^{(N)}$ can be solved as

$$s_i^{(N)} = s + (s_i^{(0)} - s)(1-\eta\lambda_i)^{2N}, \tag{53}$$

where $\eta = T/N$. Taking the limit $N \to \infty$, we have

$$\lim_{N\to\infty} s_i^{(N)} = \frac{\alpha^2}{2\lambda_i}\left(1 - e^{-2\lambda_i T}\right) \tag{54}$$

for $i = 2, 3, \dots, n$. We thus have the claim of this lemma. □

4.4 Weak convergence of Euler-Maruyama method

As previously noted, the asymptotic mean (41) is consistent with the continuous solution. The weak convergence property of the EM method [8] allows us to obtain the moments of the error at time t.

We will briefly explain the weak convergence property. Suppose a SDE with the form:

$$d\boldsymbol{x}(t) = \phi(\boldsymbol{x}(t))dt + \psi(\boldsymbol{x}(t))d\boldsymbol{b}(t). \tag{55}$$

If ϕ and ψ are bounded and Lipschitz continuous, then the finite order moment estimated by the EM method converges to the exact moment of the solution $\boldsymbol{x}(t)$ at the limit $N \to \infty$ [8]. A function $f(x)$ is said to be bounded if there exists $K > 0$ such that $\|f(x)\|^2 \le K^2(1 + \|x\|^2)$. This property is called the weak convergence property. In our case, the SDE (23) has bounded and Lipschitz continuous coefficient functions, i.e, $\phi(\boldsymbol{x}) = -\boldsymbol{L}\boldsymbol{x}$ and $\psi(\boldsymbol{x}) = \alpha$. Hence, we can employ the weak convergence property in our analysis.

Suppose $\boldsymbol{x}(t)$ is a solution of SDE (23) with the initial condition $\boldsymbol{x}(0) = \boldsymbol{c}$. Let $\boldsymbol{\mu}(t)$ be the mean vector of the residual error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \gamma\boldsymbol{1}$ and $\boldsymbol{\Sigma}(t)$ is the covariance matrix of the residual error $\boldsymbol{e}(t)$.

Theorem 1: For a positive real number $t > 0$, the mean and the covariance matrix of the residual error $\boldsymbol{e}(t)$ are given by

$$\boldsymbol{\mu}(t) = \exp(-\boldsymbol{L}t)(\boldsymbol{c} - \gamma\boldsymbol{1}) \quad (56)$$

$$\boldsymbol{\Sigma}(t) = \boldsymbol{U}\mathrm{diag}\left(\alpha^2 t, \theta_2, \theta_3, \dots, \theta_n\right)\boldsymbol{U}^T. \quad (57)$$

(Proof) Due to the weak convergence property of the EM method, the first and second moments of the error are converged to the asymptotic mean and covariance of the EM method [8], i.e.,

$$\boldsymbol{\mu}(T) = \lim_{N\to\infty} \boldsymbol{\mu}^{(N)} \quad (58)$$

$$\boldsymbol{\Sigma}(T) = \lim_{N\to\infty} \boldsymbol{\Sigma}^{(N)}, \quad (59)$$

where N and T are related by $T = \eta N$. Applying Lemmas 2 and 3 and replacing the variable T by t provide the claim of the theorem. □

4.5 Mean squared error

In the following, we assume that the initial state vector $\boldsymbol{c}$ follows Gaussian distribution $\mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$.

In this setting, $\boldsymbol{\mu}(t)$ also follows multivariate Gaussian distribution with the mean vector $\boldsymbol{0}$ and the covariance matrix $\boldsymbol{Q}(t)\boldsymbol{Q}(t)^T$ where

$$\boldsymbol{Q}(t) \equiv \exp(-\boldsymbol{L}t)\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{1}(\boldsymbol{1}^T)\right) \quad (60)$$

because $\boldsymbol{\mu}(t)$ can be rewritten as

$$\boldsymbol{\mu}(t) = \exp(-\boldsymbol{L}t)(\boldsymbol{c} - \gamma\boldsymbol{1}) = \exp(-\boldsymbol{L}t)\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{1}(\boldsymbol{1}^T)\right)\boldsymbol{c}. \quad (61)$$

By using the result of Theorem 1, we immediately have the following corollary indicating the *MSE formula*.

Corollary 1: The mean squared error (MSE)

$$\mathsf{MSE}(t) \equiv \mathsf{E}[\|\boldsymbol{x}(t) - \gamma\boldsymbol{1}\|_2^2] \quad (62)$$

is given by

$$\mathsf{MSE}(t) = \alpha^2 t + \frac{\alpha^2}{2}\sum_{i=2}^{n}\frac{1 - e^{-2\lambda_i t}}{\lambda_i} + \mathrm{tr}(\boldsymbol{Q}(t)\boldsymbol{Q}(t)^T).$$

(Proof) We can rewrite $\boldsymbol{x}(t)$ as:

$$\boldsymbol{x}(t) = \gamma\boldsymbol{1} + \boldsymbol{Q}(t)\boldsymbol{c} + \boldsymbol{w}, \quad (63)$$

where $\boldsymbol{w} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}(t))$, and $\boldsymbol{w}$ and $\boldsymbol{c}$ are independent. We thus have

$$\begin{aligned}\mathsf{MSE}(t) &= \mathrm{tr}(\boldsymbol{\Sigma}(t)) + \mathrm{tr}(\boldsymbol{Q}(t)\boldsymbol{Q}(t)^T)\\ &= \alpha^2 t + \frac{\alpha^2}{2}\sum_{i=2}^{n}\frac{1 - e^{-2\lambda_i t}}{\lambda_i} + \mathrm{tr}(\boldsymbol{Q}(t)\boldsymbol{Q}(t)^T)\end{aligned} \quad (64)$$

due to Theorem 1. □

Since the value of the term $\mathrm{tr}(\boldsymbol{Q}(t)\boldsymbol{Q}(t)^T)$ is exponentially decreasing with t, $\mathrm{tr}(\boldsymbol{\Sigma}(t))$ is dominant in $\mathsf{MSE}(t)$ for sufficiently large t. For sufficiently large t, the MSE is well approximated by the asymptotic MSE (AMSE) as

$$\mathsf{MSE}(t) \simeq \mathsf{AMSE}(t) \equiv \alpha^2 t + \frac{\alpha^2}{2}\sum_{i=2}^{n}\frac{1}{\lambda_i} \quad (65)$$

because $\mathrm{tr}(\boldsymbol{Q}(t)\boldsymbol{Q}(t)^T)$ is negligible, and $1 - e^{-2\lambda_i t}$ can be well approximated to 1. We can observe that the sum of inverse eigenvalue $\sum_{i=2}^{n}(1/\lambda_i)$ of the Laplacian matrix determines the intercept of the $\mathsf{AMSE}(t)$. In other words, the graph topology influences the stochastic error behavior through the sum of inverse eigenvalues of the Laplacian matrix.

Figure 3 presents a comparison of $\mathsf{MSE}(t)$ evaluated by the EM method (26) and the formula in (64). In this experiment, the cycle graph with 10 nodes is used. The values of $\mathsf{AMSE}(t)$ are also included in Fig. 3. We can see that the theoretical values of $\mathsf{MSE}(t)$ and estimated values by the EM method are quite close. Furthermore, the estimated values tend to approach $\mathsf{AMSE}(t)$ as t grows. This is consistent with the approximation in (65).

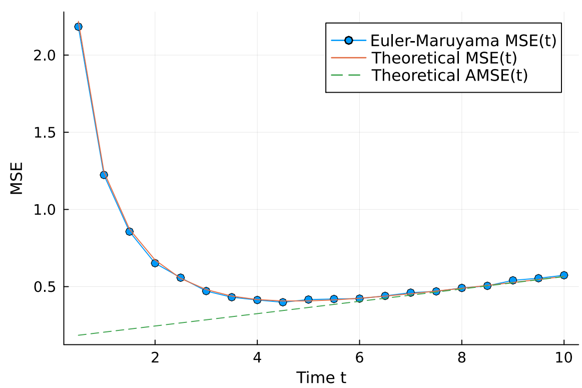

Fig. 3 Comparison of MSE: The label Euler-Maruyama represents $\mathsf{MSE}(t)$ estimated by using samples generated by the EM method. Theoretical $\mathsf{MSE}(t)$ represents the values evaluated by (64).Theoretical $\mathsf{AMSE}(t)$ represents the values of $\mathsf{AMSE}(t)$. Cycle graph with 10 nodes with $\boldsymbol{\Delta} = (2, 2, \dots, 2)$ are used. The parameter setting is as follows: $N = 250$, $T = 10$, $\alpha = 0.2$. 5000 samples are generated by the EM method for estimating $\mathsf{MSE}(t)$.

5. Conclusion

In this paper, we have formulated a noisy average consensus system through a SDE. This formulation allows for an analytical study of the stochastic dynamics of the system. We derived a formula for the evolution of covariance for the EM method. Through the weak convergence property, we have established Theorem 1 and derived a MSE formula that provides the MSE at time t. Analysis of the MSE formula reveals that the sum of inverse eigenvalues for the Laplacian matrix is the most significant factor impacting the MSE dynamics.

It is important to note that the theoretical understanding gained in this study will also provide valuable perspective on consensus-based distributed algorithms in noisy environments. The exploration of potential applications will be an open area for further studies.

Acknowledgement

This study was supported by JSPS Grant-in-Aid for Scientific Research (A) Grant Number 22H00514. A part of the results of this study was presented at the IEEE International Symposium on Information Theory (ISIT), 2022 [13]. The authors thank Prof. Masaki Ogura for letting us know the related work [6] on discrete-time average consensus systems.

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